# Resit Exam Probability Theory (WIKR-06)

11 July 2019, 09.00-12.00

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- Every exercise needs to be handed in on separate sheets, which will be collected in separate piles.
- Write your name and student number **on every sheet**.
- It is absolutely not allowed to use calculators, phones, the book, notes or any other aids.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes you are using (unless it is stated explicitly in the question that this is not needed.).
- **NOTA BENE:** using separate sheets for the different exercises and writing your name and student number on all sheets is worth 10 out of the 100 points.

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**Exercise 1, (a:10, b:10, c:10 pts).**
Let $X, Y$ be independent and Exp(1)-distributed.

(a) Show that $V := X + Y$ has the Gamma$(2, 1)$-distribution.

(b) Show that $U := \frac{X}{X+Y}$ has the uniform distribution on $[0, 1]$.

(c) Are $U, V$ independent? (Justify your answer)

**Exercise 2, (a:6, b:6, c:9, d:9 pts).**
An urn contains $n$ balls of which $r$ are red and $n-r$ blue. We draw $m$ balls from the urn (uniformly at random and) *without replacement*. Let $X$ denote the number of red balls among the balls we drew.

(a) Show that

$$\mathbb{P}(X = k) = \begin{cases} \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}} & \text{if } 0 \leq k \leq \min(m, r), \\ 0 & \text{otherwise.} \end{cases}$$

We can write $X = X_1 + \cdots + X_m$ where

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th ball is red,} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that $\mathbb{E}X_i = \frac{r}{n}$ for $i = 1, \ldots, m$. What is $\mathbb{E}X$?

(c) Determine $\mathrm{Cov}(X_i, X_j)$ for all $1 \leq i, j \leq m$. What is $\mathrm{Var}(X)$?

Now imagine we are sending the total number of balls $n$ to infinity and at the same time we send the number of red balls to infinity where $r = r(n)$ depends on $n$ via the relation $r(n) = \lfloor pn \rfloor$ with $0 < p < 1$ fixed. We keep $m$, the number of balls we draw from the (increasingly large) urn, fixed as well. Let $Y_n$ be the number of red balls when we draw $m$ balls without replacement from an urn with $n$ balls in total and $r(n) = \lfloor pn \rfloor$ red balls.

(d) Show that

$$Y_n \xrightarrow[n\to\infty]{\mathrm{d}} Y,$$

where $Y \sim \mathrm{Bin}(m, p)$.

*(Hint: You might first want to argue that it suffices to show $\mathbb{P}(Y_n = k) \to \mathbb{P}(Y = k)$ for each fixed $k$. Then you could look at the expression in (a) and divide each binomial coefficient by an appropriate power of $n$.)*

**Exercise 3, (a:5, b:5, c:5, d:5, e:5, f:5 pts)**

(a) State and prove Markov's inequality. (For the proof, you may restrict yourself to the discrete case.)

(b) State and prove Chebyschev's inequality.

A function $\varphi : \mathbb{R} \to \mathbb{R}$ is *convex* if

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y),$$

for all $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$.

(c) Let $Z$ be a random variable with $-1 \leq Z \leq 1$ and $\mathbb{E}Z = 0$, and let $\varphi$ be a convex function. Show that $\mathbb{E}\varphi(Z) \leq \frac{1}{2}(\varphi(-1) + \varphi(1))$, and deduce that in particular the moment generating function of $Z$ satisfies $\mathbb{E}e^{tZ} \leq \cosh(t) = \frac{1}{2}(e^{-t} + e^{t})$.

*(Hint: write $z \in [-1,1]$ as $z = \lambda \cdot (-1) + (1-\lambda) \cdot (+1)$, where $\lambda = \lambda(z) \in [0,1]$ depends on $z$. Take expectations on both sides of the inequality $\varphi(z) \leq \lambda(z)\varphi(-1) + (1-\lambda(z))\varphi(1)$. You may use without proof that $z \mapsto e^{tz}$ is convex.)*

Now suppose that $X$ is $\text{Bin}(n,p)$. So we can write $X = X_1 + \cdots + X_n$ with the $X_i$ i.i.d. $\text{Be}(p)$-distributed. Write $Y_i = X_i - \mathbb{E}X_i, Y = \sum_{i=1}^n Y_i = X - \mathbb{E}X$.

(d) Show that $\mathbb{P}(X - \mathbb{E}X > \lambda) \leq e^{-t\lambda} \cdot \mathbb{E}e^{tY}$, for every $t > 0$.

*(Hint: use Markov.)*

(e) Show that $\mathbb{E}e^{tY} \leq e^{nt^2/2}$ and deduce that $\mathbb{P}(X - \mathbb{E}X > \lambda) \leq e^{-\lambda^2/(2n)}$.

*(Hint: use (c), that $\cosh(t) \leq e^{t^2/2}$, and your knowledge of moment generating functions.)*

(f) Show that $\mathbb{P}(|X - \mathbb{E}X| > \lambda) \leq 2e^{-\lambda^2/(2n)}$.

*Closing remark : if you managed to solve (f) then you've just proved the Chernoff inequality for the binomial, which often gives a much better (=smaller) bound on the probability that $|X - \mathbb{E}X| > \lambda$ than the Chebyschev inequality.*